The Interval-Valued Choquet Integral Based on Admissible Permutations

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***Abstract*—Aggregation or fusion of interval data is not a trivial task, since the necessity of arranging data arises in many aggregation functions, such as OWA operators or the Choquet integral. Some arranging procedures have been given to solve this problem, but they need certain parameters to be set. In order to solve this problem, in this work we propose the concept of an admissible permutation of intervals. Based on this concept, which avoids any parameter selection, we propose a new approach for the interval-valued Choquet integral that takes into account every possible permutation fitting to the considered ordinal structure of data. Finally, a consensus among all the permutations is constructed.**

***Index Terms*—Information Fusion, Choquet Integral, Interval-Valued Choquet Integral, Admissible Order, Admissible Permutation.**

I. Introduction

AGGREGATION techniques are nowadays a very important tool, since the need to fuse several values (coming from different inputs) into a single one is a key step that appears in almost every application [1], [2], [3]. For example, in decision making problems, the inputs could be experts opinions or evaluations of criteria; in sensor fusing, data coming from different sensors must be combined so as to reduce uncertainty of data [4]; in image processing, the fusion of adjacent pixels or the fusion of pixels coming from different images is the foundation of image filtering, noise reconstruction, reduction, stereo vision, etc. [5], [6].

Extensions of fuzzy sets, such as interval-valued fuzzy sets, Atanassov's intuitionistic fuzzy sets, type-2 fuzzy sets, among others, have become very popular in many applications in order to deal with the uncertainty inherent to data [7], [8], [9], [10]. In this work we focus on intervals as the source of information that must be aggregated since, with moderate complexity, they allow to model uncertainty adequately. For example, information coming from sensors can be modeled by intervals when it takes inherent accuracy of instrumentation. In decision making, intervals are adequate when experts may not be able of assigning an exact opinion or numerical evaluation of an alternative. Moreover, an interval of labels, such as "something in between good and very good", could be more appropriate. In image processing, the intensity of a specific pixel may not be too informative. However, applications where the pixel intensity is transformed into an interval that take also into account neighbors' information have been proven to obtain better results in some applications [11], [12].

In order to extend aggregation functions such as the Choquet integral or OWA operators to the interval setting, one of the main difficulties is the arrangement of input data in a decreasing (or in an increasing) way. Genuine partial ordering of intervals excludes the possibility of a direct use of formulas for the (discrete) Choquet integral, where the weights of single inputs are derived from a considered measure by means of some permutation relevant to the ordinal structure of the real data to be aggregated. One way to solve this problem is related to Auman's approach to integration of set-valued functions [13]. This approach neglects the ordinal structure of the considered interval data, and, instead of, it deals separately with lower and upper bounds. An alternative approach, based on the concept of admissible order, was proposed in [14]. This approach performs the arrangement of data using a certain admissible order. Then, a new problem is opened - the choice of an appropriate admissible order. Our aim is to introduce a new method, based purely on the ordinal structure of interval data to be aggregated, thus preserving the original idea of Choquet [15].

To do so, in this paper we propose the concept of an admissible permutation of a set of intervals. This concept allows us to know how many different arrangements can be obtained by admissible orders. Then, we propose a new interval-valued Choquet integral that considers every possible arrangement (instead of single one) and constructs a consensus result among them.

Finally, we illustrate the use of admissible permutations in a decision-making problem based on interval preference relations.

This paper is organized as follows. In Section II we recall the main concepts used along this paper. In Section III we explore and analyze previous approaches to the interval Choquet integral. In Sections IV and V we introduce the concept of admissible permutation and interval-valued Choquet integral based on admissible permutations, respectively. We finish with an illustrative example of decision-making in Section VI and conclusions in Section VII.

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II. Preliminaries

We start recalling the concept of an aggregation function.

Definition 1: [16] Let $(L, \preceq)$ be a bounded partially ordered set with a least element 0_L and a greatest element 1_L. A mapping $M : L^n \to L$ is an n-ary aggregation function if it satisfies the properties:

(i) $M(0_L, \ldots, 0_L) = 0_L$ and $M(1_L, \ldots, 1_L) = 1_L$;

(ii) it is increasing in each argument, i.e., for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in L^n$, $M(x_1, \ldots, x_n) \preceq M(y_1, \ldots, y_n)$ whenever $x_1 \preceq y_1, \ldots, x_n \preceq y_n$.

Observe that if L is the unit interval equipped with the standard ordering of reals, $L = [0, 1]$, then we obtain the usual definition of aggregation function [1], [2], [3].

In this paper we deal with aggregation of intervals. Therefore, we take $L = L([0, 1])$ as the set of all closed subintervals in $[0, 1]$:

$$L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | 0 \leq \underline{x} \leq \overline{x} \leq 1\}.$$

Note that $L([0, 1])$ is a partially ordered set with respect to the order relation $\leq_L$ defined in the following way: for any $\mathbf{x}, \mathbf{y} \in L([0, 1])$,

$$\mathbf{x} \leq_L \mathbf{y} \text{ if and only if } \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}.$$

In fact, $(L([0, 1]), \leq_L)$ is a complete lattice where the least element is $0_L = [0, 0]$ and the greatest element is $1_L = [1, 1]$ ([17]). In this lattice, the infimum and supremum of any two elements are given, respectively, by

$$\begin{aligned} \mathbf{x}_1 \wedge \mathbf{x}_2 &= [\min(\underline{x}_1, \underline{x}_2), \min(\overline{x}_1, \overline{x}_2)] \\ \mathbf{x}_1 \vee \mathbf{x}_2 &= [\max(\underline{x}_1, \underline{x}_2), \max(\overline{x}_1, \overline{x}_2)]. \end{aligned}$$

Example 1: The following are examples of aggregation functions defined on $(L([0, 1]), \leq_L)$:

$$\mathbf{M}_{arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \left[\frac{1}{n} \sum_{i=1}^{n} \underline{x}_i, \frac{1}{n} \sum_{i=1}^{n} \overline{x}_i \right];$$

$$\mathbf{M}_{min}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \left[\min_{i=1,\ldots,n} \underline{x}_i, \min_{i=1,\ldots,n} \overline{x}_i \right];$$

$$\mathbf{M}_{max}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \left[\max_{i=1,\ldots,n} \underline{x}_i, \max_{i=1,\ldots,n} \overline{x}_i \right];$$

$$\mathbf{M}_{geom,arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \left[\prod_{i=1}^{n} \underline{x}_i, \sum_{i=1}^{n} \overline{x}_i \right];$$

Remark 1: Observe that $\mathbf{M}_{min}$ and $\mathbf{M}_{max}$ coincide with infimum and supremum of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively.

A. Admissible orders on $L([0, 1])$

The order $\leq_L$ defined for intervals in subsection II is a partial order on $L([0, 1])$. This means that it is not always possible to compare or arrange two arbitrary intervals. However, several aggregation functions defined on $[0, 1]$, such as OWA operators or the Choquet and Sugeno integrals, are based on the arrangement of the inputs. Therefore, if we want to extend these aggregation functions to $L([0, 1])$, we need to consider the problem associated with the partial order.

In order to solve the problem, in [18] the concept of admissible order was introduced. Admissible orders are, in fact, linear orders on $L([0, 1])$ refining the partial order $\leq_L$.

Definition 2: [18] Consider the partially order set $(L([0, 1]), \preceq)$. The order $\preceq$ on $L([0, 1])$ is called an admissible order if

(i) $\preceq$ is a linear order on $L([0, 1])$;

(ii) for all $\mathbf{x}, \mathbf{y} \in L([0, 1])$, $\mathbf{x} \preceq \mathbf{y}$ whenever $\mathbf{x} \leq_L \mathbf{y}$.

Example 2: The following are examples of admissible orders:

(i) $\mathbf{x} \preceq_{Lex1} \mathbf{y}$ (usual lexicographic order in $\mathbb{R}^2$ transformed onto $L([0, 1])$) if and only if $\underline{x} < \underline{y}$ or ($\underline{x} = \underline{y}$ and $\overline{x} \leq \overline{y}$);

(ii) $\mathbf{x} \preceq_{Lex2} \mathbf{y}$ if and only if $\overline{x} < \overline{y}$ or ($\overline{x} = \overline{y}$ and $\underline{x} \leq \underline{y}$);

(iii) $\mathbf{x} \preceq_{XY} \mathbf{y}$ (Xu-Yager order given in [19]) if and only if $\underline{x} + \overline{x} < \underline{y} + \overline{y}$ or ($\underline{x} + \overline{x} = \underline{y} + \overline{y}$ and $\overline{y} - \underline{y} \leq \overline{x} - \underline{x}$).

Admissible orders can be generated by means of two aggregation functions on $[0, 1]$ fulfilling certain conditions. Let $K([0, 1]) = \{(\underline{x}, \overline{x}) \in [0, 1]^2 | \underline{x} \leq \overline{x}\}$.

Proposition 1: [18] Let $A, B : [0, 1]^2 \to [0, 1]$ be two aggregation functions on $[0, 1]$ such that, for all $(x, y), (u, v) \in K([0, 1])$, the equalities $A(x, y) = A(u, v)$ and $B(x, y) = B(u, v)$ can hold simultaneously only if $(x, y) = (u, v)$. Define the relation $\preceq_{\mathrm{A,B}}$ on $L([0, 1])$ by $\mathbf{x} \preceq_{\mathrm{A,B}} \mathbf{y}$ if and only if

$$A(\underline{x}, \overline{x}) \quad < \quad A(\underline{y}, \overline{y}) \quad \text{or} \tag{1}$$

$$\left(A(\underline{x}, \overline{x}) = A(\underline{y}, \overline{y}) \quad \text{and} \quad B(\underline{x}, \overline{x}) \leq B(\underline{y}, \overline{y})\right). \tag{2}$$

Then, $\preceq_{\mathrm{A,B}}$ is an admissible order on $L([0, 1])$.

B. Fuzzy measures and discrete Choquet integral

Prior to the definition of the Choquet integral, we recall the concept of a fuzzy measure (see [20], [21]).

Definition 3: Let $X = \{1, \ldots, n\}$. A fuzzy measure over X is a mapping $m : 2^X \to [0, 1]$ such that

(i) $m(\emptyset) = 0$ and $m(X) = 1$;

(ii) If $E \subset F$, then $m(E) \leq m(F)$.

Example 3:

- The bottom fuzzy measure is defined by

$$m_*(E) = \begin{cases} 1 & \text{if } E = X; \\ 0 & \text{otherwise.} \end{cases}$$

  Observe that for any other fuzzy measure m over X, it holds that $m_*(E) \leq m(E)$, for every $E \subseteq X$.

- The top fuzzy measure is defined by

$$m^*(E) = \begin{cases} 0 & \text{if } E = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

  Observe that for any other fuzzy measure m over X, it holds that $m(E) \leq m^*(E)$ for every $E \subseteq X$.

Definition 4: [15] Let $m : 2^X \to [0, 1]$ be a fuzzy measure. The discrete Choquet integral of $x_1, \ldots, x_n \in [0, 1]$ with respect to m is defined by

$$\begin{aligned} C_m(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_{\sigma(i)} (m(\{\sigma(i), \ldots, \sigma(n)\}) - \\ m(\{\sigma(i+1), \ldots, \sigma(n)\})) \end{aligned} \tag{3}$$

where $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation such that $x_{\sigma(1)} \le \cdots \le x_{\sigma(n)}$ and $\{x_{\sigma(n+1)}, x_{\sigma(n)}\} = \emptyset$, by convention.

An alternative definition of the discrete Choquet integral was shown in [22], namely

$$C_m(x_1,\ldots,x_n) = \sup\left\{\sum_{i=1}^n r_i m(E_i) | (r_1,\ldots,r_n) \in [0,1]^n, \right.$$
$$\left. (E_i)_{i=1}^n \text{ is a chain in } X \text{ and } \sum_{i=1}^n r_i 1_{E_i} \le (x_1,\ldots,x_n)\right\}. \tag{4}$$

where 1_{E_i} is the characteristic function of the set E_i. Note that the supremum is attained, i.e. it can be replaced by the max operator.

III. Different approaches to the interval-valued Choquet integral

In the literature we find several approaches that extend the discrete Choquet integral to the interval setting. In this section we recall some of them and we show some new results. Both the advantages and disadvantages of each approach are analyzed.

A. Interval-valued Choquet integral based on Auman's approach and integral decomposition approach

The first approach of interval-valued Choquet integral is based on Aumann's integral definition for set-valued functions [13].

Definition 5: Let $m : 2^X \to [0,1]$ be a fuzzy measure and let $\mathbf{x}_1,\ldots,\mathbf{x}_n \in L([0,1])$. The discrete interval-valued Choquet integral $\mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ with respect to m is given by

$$\mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \{C_m(x_1,\ldots,x_n) | x_i \in \mathbf{x}_i\}. \tag{5}$$

Taking into account the monotonicity and continuity of the discrete Choquet integral, it follows that

$$\mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) = [C_m(\underline{x}_1,\ldots,\underline{x}_n), C_m(\overline{x}_1,\ldots,\overline{x}_n)].$$

This means that the interval-valued Choquet integral based on Aumann's approach is given by the set of all Choquet integrals (defined on $[0,1]$) applied to every possible n-tuple of real numbers each of them within the corresponding interval.

Remark 2: Notice that $\mathbf{C}_{m_*} = \mathbf{M}_{\min}$ and $\mathbf{C}_{m^*} = \mathbf{M}_{\max}$.

Another approach to the discrete interval-valued Choquet integral is studied in [23], [22] as a particular case of *decomposition integral*, in which the set system is a chain of sets. We extend this approach to the interval setting $L([0,1])$ and we prove that, even though both approaches have different inspiration, they yield the same result.

Definition 6: Let $m : 2^X \to [0,1]$ be a fuzzy measure and let $\mathbf{x}_1,\ldots,\mathbf{x}_n \in L([0,1])$. The interval-valued Choquet integral $\mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ based on the decomposition integral is given by

$$\mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sup\left\{\sum_{i=1}^n \mathbf{r}_i\, m(E_i) | (\mathbf{r}_1,\ldots,\mathbf{r}_n) \in L([0,1])^n, \right.$$
$$\left. (E_i)_{i=1}^n \text{ is a chain in } 2^X \text{and } \sum_{i=1}^n \mathbf{r}_i\, 1_{E_i} \le_L (\mathbf{x}_1,\ldots,\mathbf{x}_n)\right\} \tag{6}$$

where 1_{E_i} is the characteristic function of the set E_i.

Proposition 2: Let $m : 2^X \to [0,1]$ be a fuzzy measure. Then it holds that for any $\mathbf{x}_1,\ldots,\mathbf{x}_n \in L([0,1])$

$$\mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

Proof: Consider any situation in which $\sum_{i=1}^n \mathbf{r}_i\, 1_{E_i} \le_L (\mathbf{x}_1,\ldots,\mathbf{x}_n)$. Then, $\sum_{i=1}^n \underline{r}_i 1_{E_i} \le (\underline{x}_i,\ldots,\underline{x}_n)$ and, due to Eq. 4, $\sum_{i=1}^n \underline{r}_i m(E_i) \le C_m(\underline{x}_i,\ldots,\underline{x}_n)$. Thus, $\underline{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) \le C_m(\underline{x}_1,\ldots,\underline{x}_n)$. Similarly, for the upper bounds we have that $I_m(\overline{x}_1,\ldots,\overline{x}_n) \le C_m(\overline{x}_1,\ldots,\overline{x}_n)$ and hence

$$\mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) \le_L \mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

Based in Eqs. 3 and 4, as already mentioned at the end of previous section, it can be shown that there is a chain (E_i) in 2^X and constants $c_1,\ldots,c_n \in [0,1]$ such that $\sum_{i=1}^n c_i 1_{E_i} = (\underline{x}_1,\ldots,\underline{x}_n)$ and $C_m(\underline{x}_1,\ldots,\underline{x}_n) = \sum_{i=1}^n c_i m(E_i)$. Putting $\mathbf{r}_i = [c_i, c_i]$, obviously $\sum_{i=1}^n \mathbf{r}_i\, 1_{E_i} \le_L (\mathbf{x}_1,\ldots,\mathbf{x}_n)$ and, thus, $C_m(\underline{x}_1,\ldots,\underline{x}_n) \le \sum_{i=1}^n c_i m(E_i) \le \underline{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. Similarly, one can show that $C_m(\overline{x}_1,\ldots,\overline{x}_n) \le \overline{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ and hence

$$\mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) \le_L \mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

Summarizing, it holds that

$$\mathbf{I}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{C}_m(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

■

Remark 3: Let us denote by

$$A = \left\{\sum_{i=1}^n \mathbf{r}_i\, m(E_i) | (\mathbf{r}_1,\ldots,\mathbf{r}_n) \in L([0,1])^n, \right.$$
$$\left. (E_i)_{i=1}^n \text{ is a chain in } 2^X \text{and } \sum_{i=1}^n \mathbf{r}_i\, 1_{E_i} \le_L (\mathbf{x}_1,\ldots,\mathbf{x}_n)\right\}$$

Notice that, in general $\sup\{A\} \notin A$.

Example 4: Let $\mathbf{x}_1 = [0,1], \mathbf{x}_2 = [1/2,1/2]$ and let $m : 2^{\{1,2\}} \to [0,1]$ be given by $m(\{1\}) = a, m(\{2\}) = b$, with $a, b \in [0,1]$. We have two possible chains, namely $E_1 = \{1,2\}, E_2 = \{1\}$ and $E_1 = \{1,2\}, E_2 = \{2\}$. For the first case we have that, according to the restriction $\mathbf{r}_1\, 1_{E_1} + \mathbf{r}_2\, 1_{E_2} \le_L (\mathbf{x}_1, \mathbf{x}_2)$, we have that $\mathbf{r}_1 + \mathbf{r}_2 \le_L [0,1]$ and $\mathbf{r}_1 \le_L [1/2,1/2]$, that implies $\underline{r}_1 = \underline{r}_2 = 0, \overline{r}_1 \le 1/2$ and $\overline{r}_1 + \overline{r}_2 \le 1$. Then, $\sup\{\mathbf{r}_1\, m(E_1) + \mathbf{r}_2\, m(E_2)\} = [0, \underline{r}_1 + a\underline{r}_2] = [0,(a+1)/2]$ is achieved with $\overline{r}_1 = \overline{r}_2 = 1/2$. In the second case, we have that $\mathbf{r}_1 \le_L [0,1]$ and $\mathbf{r}_1 + \mathbf{r}_2 \le_L [1/2,1/2]$, which implies $\underline{r}_1 = 0, \overline{r}_1 + \overline{r}_2 \le 1/2$. Then, $\sup\{\mathbf{r}_1\, m(E_1) + \mathbf{r}_2\, m(E_2)\} = [\underline{r}_1 + b\underline{r}_2, \overline{r}_1 + b\overline{r}_2] = [b/2, b/2]$ is achieved with $\mathbf{r}_1 = [0,0], \mathbf{r}_2 = [1/2,1/2]$. Finally, $\mathbf{I}_m(\mathbf{x}_1, \mathbf{x}_2) = [b/2,(a+1)/2]$. It is easy to see that whenever $b > 0$, then $\sup\{A\} \notin A$.

Taking now Auman's approach, we have that $\mathbf{C}_m(\mathbf{x}_1, \mathbf{x}_2) = [C_m(0,1/2), C_m(1,1/2)] = [b/2,(a+1)/2]$.

B. Interval-valued Choquet integrals induced by admissible orders

The previous approach to the interval-valued Choquet integral did not consider the ordinal structure of data. In [14], a new approach to the interval-valued Choquet integral was given. The originality of this new approach is based on the arrangement of interval inputs using any admissible order. In this sense, each admissible order generates a specific interval-valued Choquet integral.

Definition 7: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$ and let $m : 2^X \to [0,1]$ be a fuzzy measure. The discrete interval-valued Choquet integral with respect to an admissible order $\preceq_{\mathrm{A,B}}$ with notation $\mathbf{C}_m^{\preceq_{\mathrm{A,B}}}$ is given by

$$\mathbf{C}_m^{\preceq_{\mathrm{A,B}}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} \mathbf{x}_{\sigma_{A,B}(i)} (m(\{\sigma_{A,B}(i), \ldots, \sigma_{A,B}(n)\}) - m(\{\sigma_{A,B}(i+1), \ldots, \sigma_{A,B}(n)\}))$$

where $\sigma_{A,B} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that

$$\mathbf{x}_{\sigma_{A,B}(1)} \preceq_{\mathrm{A,B}} \cdots \preceq_{\mathrm{A,B}} \mathbf{x}_{\sigma_{A,B}(n)}$$

and $\{\sigma_{A,B}(n), \sigma_{A,B}(n+1)\} = \emptyset$ by convention.

One of the main advantages of this approach is the fact that $\mathbf{C}_m^{\preceq_{\mathrm{A,B}}}$ generalizes the usual discrete Choquet integral defined on $[0,1]$, since

$$\mathbf{C}_m^{\preceq_{\mathrm{A,B}}}([x_1, x_1], \ldots, [x_n, x_n]) = [C_m(x_1, \ldots, x_n), C_m(x_1, \ldots, x_n)].$$

Besides, if every input is comparable with respect to the partial order $\leq_L$, it obtains the same result than Aumann's approach, as it is proven in [14].

Moreover, comparing this approach with Auman's, we notice that interval inputs are considered as a whole, and lower/upper bounds are not split in the calculation of the integral. This means that the ordinal structure of data, which is determined by the admissible order, is taken into account for obtaining the fused value. This fact makes the interval-valued Choquet integral with respect to an admissible order more adequate than other approaches in problems such as multi-criteria decision-making. In this kind of problems, there usually exists a close relation between the fuzzy measure (that may represent interaction between criteria) and the interval inputs (that may represent the degree of satisfaction of criteria).

On the other side, some drawbacks of this proposal arise. First, the number of admissible orders is infinite and many of them are equivalent. Second, but related, it is very difficult to know which order must be used for an specific application.

In the following two sections we solve these problems using the concept of admissible permutations and defining an interval-valued Choquet integral based on every admissible permutation.

IV. The concept of admissible permutation and its relation with admissible orders

As we have mentioned in the previous section, there exist infinitely many admissible orders. In fact, as it is proven in Proposition 3.8 of [18], many admissible orders are equivalent and yield in the same arrangement for a fixed vector of elements. Moreover, when we order a finite number of inputs (intervals), the number of possible arrangements is again finite, even if the number of admissible orders is infinite.

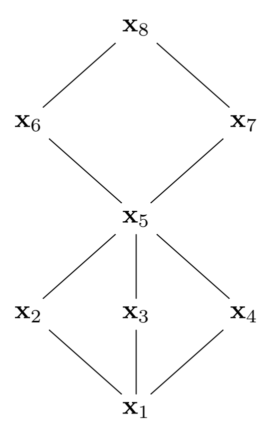
It is also important to mention that, given n intervals, the total $n!$ of its potential permutations need not be allowed. This is due to the fact that admissible orders are refinements of the usual partial order $\leq_L$. Therefore, given $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$ and a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, if $\mathbf{x}_i <_L \mathbf{x}_j$, then we should ensure that $\sigma^{-1}(i) < \sigma^{-1}(j)$. Moreover, if $\mathbf{x}_i = \mathbf{x}_j$, then for any k between $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$, we should ensure that $\mathbf{x}_i = \mathbf{x}_{\sigma(k)} = \mathbf{x}_j$. In order to clarify this idea, we introduce the concept of an admissible permutation of data.

Definition 8: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$. A permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is said to be an admissible permutation with respect to the partial order $\leq_L$ if

(i) for every $\mathbf{x}_i <_L \mathbf{x}_j$, we have that $\sigma^{-1}(i) < \sigma^{-1}(j)$ and
(ii) for each $\mathbf{x}_i$, the set $\{\sigma^{-1}(j) | j \in \{1, \ldots, n\}$ with $\mathbf{x}_i = \mathbf{x}_j\}$ is an interval in $\mathbb{N}$.

The first property of Definition 8 ensures that the permutation does not alter the order of comparable intervals with the partial order. The second property ensures that equal intervals are ordered consecutively.

Example 5: Consider the following set of inputs: $\mathbf{x}_1 = [0,0]$, $\mathbf{x}_2 = [0.1, 0.3]$, $\mathbf{x}_3 = [0.15, 0.25]$, $\mathbf{x}_4 = [0.2, 0.2]$, $\mathbf{x}_5 = [0.2, 0.4]$, $\mathbf{x}_6 = [0.3, 0.7]$, $\mathbf{x}_7 = [0.4, 0.5]$ and $\mathbf{x}_8 = [0.6, 0.8]$. We show the Hasse diagram of these intervals in the lattice $(L([0,1]), \leq_L)$.



It is easy to see that there exist 12 admissible permutations of these intervals. Since $\mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ are incomparable, there exist $3! = 6$ permutations of $\{2, 3, 4\}$. Now, since $\mathbf{x}_6$ and $\mathbf{x}_7$ are also incomparable, we have $2! = 2$ permutations of $\{6, 7\}$. Finally, the number of admissible permutations is given by $3!2! = 12$, drastically reducing the number $8! = 40320$ of all possible permutations.

As we have mentioned, many different admissible orders may induce the same arrangement of a set of intervals. Moreover, it is not easy to know *a priori* which admissible order should be used in order to obtain a certain arrangement. For these reasons, the importance of admissible permutations lies in the fact that, for a particular input vector, we can omit the choice of the admissible order. This is proven in the next theorem, in which we prove that any admissible permutation is induced by a certain admissible order.

Theorem 1: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$. The permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is an admissible permutation of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ if and only if there exists an admissible order $\preceq_{\mathrm{A,B}}$ such that

$$\mathbf{x}_{\sigma(1)} \preceq_{\mathrm{A,B}} \cdots \preceq_{\mathrm{A,B}} \mathbf{x}_{\sigma(n)}\,.$$

Proof: Sufficiency is direct, since $\preceq_{\mathrm{A,B}}$ refines the partial order $\leq_L$. To prove necessity, we must find a pair of admissible aggregation functions (A, B). Let us start by constructing, for each $\mathbf{x}_i$, the set $E_i = \{j \in \{1, \ldots, n\} \,|\, \mathbf{x}_i = \mathbf{x}_j\}$. Now, define $\tilde{A}(0,0) = 0$, $\tilde{A}(1,1) = 1$ and, for each $\mathbf{x}_i \notin \{[0,0],[1,1]\}$,

$$\tilde{A}(\underline{x}_i, \overline{x}_i) = \frac{\min\{\sigma^{-1}(j) | j \in E_i\}}{n+1}.$$

Observe that, for a function $\tilde{A} : [0,1]^2 \to [0,1]$, the values $\tilde{A}(\underline{x}_i, \overline{x}_i)$ for every $i \in \{1, \ldots, n\}$, $\tilde{A}(0,0)$ and $\tilde{A}(1,1)$ are compatible with the constraints for aggregation functions, i.e. boundary conditions and monotonicity. Thus, we can extend $\tilde{A}$ to the whole domain $[0,1]^2$, for every $(u,v) \in [0,1]^2$, by taking

$$\tilde{A}(u,v) = \begin{cases} \min\{\tilde{A}(\underline{x}_i, \overline{x}_i) | i \in E_{(u,v)}\} & \text{if } E_{(u,v)} \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

where $E_{(u,v)} = \{j \in \{1, \ldots, n\} | (u,v) \leq (\underline{x}_j, \overline{x}_j)\}$. Now we have to prove the monotonicity of $\tilde{A}$ in the whole domain. Take arbitrary $u, u', v, v' \in [0,1]$ with $u \leq u'$ and $v \leq v'$. If $E_{(u',v')} = \emptyset$ then $\tilde{A}(u,v) \leq \tilde{A}(u',v')$. Otherwise, whenever $i \in E_{(u',v')}$ it holds that $i \in E_{(u,v)}$ since $u \leq u' \leq \underline{x}_i$ and $v \leq v' \leq \overline{x}_i$. However, the contrary is not true, so $E_{(u',v')} \subseteq E_{(u,v)}$ and, therefore, $\tilde{A}(u,v) \leq \tilde{A}(u',v')$. Finally, define the mappings $A, B : [0,1]^2 \to [0,1]$ in the following way:

$$\begin{aligned} A(x,y) &= \frac{\tilde{A}(x,y) + \frac{xy}{n+1}}{1 + \frac{1}{n+1}} \\ B(x,y) &= \frac{x^2 + y}{2}. \end{aligned}$$

It is clear that A, B are aggregation functions, due to the increasing monotonicity of $\tilde{A}$. Now, let us prove that they form an admissible pair of aggregation functions. Suppose any $(x,y), (u,v) \in K([0,1])$ such that $A(x,y) = A(u,v)$ and suppose that $\tilde{A}(x,y) < \tilde{A}(u,v)$. Having in mind the construction method of $\tilde{A}$, we have that $\tilde{A}(u,v) - \tilde{A}(x,y) \geq \frac{1}{n+1}$, so it implies that $xy \geq 1 + uv$. But this is only possible if $x = y = 1$ and we have a contradiction since $\tilde{A}(x,y) = 1 < \tilde{A}(u,v)$. The same reasoning can be done with the supposition $\tilde{A}(u,v) < \tilde{A}(x,y)$ and, therefore, the assumption $A(x,y) = A(u,v)$ implies that $\tilde{A}(x,y) = \tilde{A}(u,v)$ and, therefore, $xy = uv$. Now, if we consider the second equality $B(x,y) = B(u,v)$, we have that $x^2 + y = u^2 + v$. It is only a matter of calculation to prove that, if $xy = uv$ and $x^2 + y = u^2 + v$ hold simultaneously, then necessarily $x = u$ and $y = v$, so the pair (A, B) is admissible and $\preceq_{\mathrm{A,B}}$ is an admissible order. Moreover, it satisfies

$$\mathbf{x}_{\sigma(1)} \preceq_{\mathrm{A,B}} \cdots \preceq_{\mathrm{A,B}} \mathbf{x}_{\sigma(n)}$$

which proves the necessity of the theorem. ■

It is important to mention that both concepts, admissible order and admissible permutation, are not equivalent, since the definition of admissible permutation depends on each specific input vector. For a fixed input vector, Theorem 1 shows, on the one hand, that each admissible order induces an admissible permutation and, on the other hand, that each admissible permutation can be obtained by some admissible order. However, recall that we can have many admissible permutations of a given vector $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in L([0,1])^n$. For each admissible permutation σ, we can obtain a fused value C_m^{σ} which is given by

$$\mathbf{C}_m^{\sigma}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} \mathbf{x}_{\sigma(i)} (m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})).$$

Notice that, if $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in L([0,1])$ have p admissible permutations $\sigma_1, \ldots, \sigma_p$, with $1 \leq p \leq n!$, then we can obtain p fused values $\mathbf{C}_m^{\sigma_1}, \ldots, \mathbf{C}_m^{\sigma_p}$.

V. The interval-valued Choquet integral with respect to every admissible permutation

In the previous section we have proposed the concept of admissible permutation and we have seen that, given a set of intervals, there exists a finite number of admissible permutations (that may vary from 1 to $n!$) for sorting them. Even though we have solved the problem of having infinitely many admissible orders, we still have a second problem: which admissible permutation should be used. In fact, given a vector $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in L([0,1])$, we can potentially obtain as many fused values as the number of admissible permutations of the input vector.

To solve this problem, our proposal consists in defining a new interval-valued Choquet integral whose result is given as a consensus or agreed value obtained considering each individual admissible permutation. We propose to obtain this consensus by calculating the arithmetic mean of each fused value $\mathbf{C}_m^{\sigma}$, as it is explained in the following definition.

Definition 9: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$ and let $m : 2^n \to [0,1]$ be a fuzzy measure. Let $\sigma_1, \ldots, \sigma_p : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the set of all admissible permutations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. The consensus (via arithmetic mean) interval-valued Choquet integral $\mathbf{C}_m^{arith}$ with respect to m is given, respectively, by

$$\mathbf{C}_m^{arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{M}_{arith}(\mathbf{C}_m^{\sigma_1}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, \mathbf{C}_m^{\sigma_p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)). \quad (7)$$

Remark 4: Although in this paper we focus on the consensus interval-valued Choquet integral, it is interesting to explore the use of the extreme functions, such as the supremum and infimum operations instead of the arithmetic mean, namely $\mathbf{C}_m^{inf}, \mathbf{C}_m^{sup} : L([0,1])^n \to L([0,1])$ given by

$$\mathbf{C}_m^{inf}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \inf\{\mathbf{C}_m^{\sigma_1}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, \mathbf{C}_m^{\sigma_p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\}, \quad (8)$$

$$\mathbf{C}_m^{sup}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sup\{\mathbf{C}_m^{\sigma_1}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \ldots, \mathbf{C}_m^{\sigma_p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\}. \quad (9)$$

In this sense, $\mathbf{C}_m^{inf}$ can be considered as a pessimistic evaluation, while $\mathbf{C}_m^{sup}$ can be considered as an optimistic one. The choice of a decision maker between these possible aggregation methods depends either on his/her attitude or on the nature of the problem to be solved.

Example 6: Consider the following set of intervals that must be aggregated using the interval-valued Choquet integral:

$$\begin{aligned}\mathbf{x}_1 &= [0.1, 1],\\ \mathbf{x}_2 &= [0.3, 0.9],\\ \mathbf{x}_3 &= [0.4, 0.4].\end{aligned}$$

Let m be a fuzzy measure given by

$$\begin{array}{lll} m(\{\emptyset\}) = 0, & & \\ m(\{1\}) = 0.1, & m(\{2\}) = 0.5, & m(\{3\}) = 0.2,\\ m(\{1,2\}) = 0.5, & m(\{1,3\}) = 0.3, & m(\{2,3\}) = 0.9,\\ m(\{1,2,3\}) = 1. & & \end{array}$$

First, recall that the result of Aumann's approach is given by

$$\begin{aligned}\mathbf{C}_m(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) =&[C_m(0.1, 0.3, 0.4), C_m(1, 0.9, 0.4)] =\\ &[0.3, 0.66]\end{aligned}$$

Now, let us calculate the interval-valued Choquet integral with respect to every admissible permutation. Since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are not comparable (by means of $\leq_L$) there exist six admissible permutations $\sigma_1, \dots, \sigma_6$, that are given in the following table

$$\begin{aligned}\sigma_1 &= (1, 2, 3)\\ \sigma_2 &= (1, 3, 2)\\ \sigma_3 &= (2, 1, 3)\\ \sigma_4 &= (2, 3, 1)\\ \sigma_5 &= (3, 1, 2)\\ \sigma_6 &= (3, 2, 1)\end{aligned}$$

Then, we have that

$$\begin{array}{rcl} \mathbf{C}_m^{\sigma_1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.30, 0.81]\\ \mathbf{C}_m^{\sigma_2}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.32, 0.71]\\ \mathbf{C}_m^{\sigma_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.30, 0.81]\\ \mathbf{C}_m^{\sigma_4}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.30, 0.81]\\ \mathbf{C}_m^{\sigma_5}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.35, 0.65]\\ \mathbf{C}_m^{\sigma_6}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &=& [0.33, 0.66].\end{array}$$

Finally, we can calculate the consensus interval-valued Choquet integral by using the arithmetic mean of the previous values

$$\mathbf{C}_m^{arith}([0.1, 1], [0.3, 0.9], [0.4, 0.4]) = [0.317, 0.742].$$

Observe that $\mathbf{C}_m \leq_L \mathbf{C}_m^{arith}$ for this example and, specifically, the upper bounds are much more distant. This is probably explained by how Auman's approach uses two different permutations for the lower and upper bounds. In the case of the upperbounds, we have that $\overline{x}_3 < \overline{x_2} < \overline{x_1}$ and, therefore,

$$\begin{aligned}\overline{C}_m =&\overline{x}_3(m(\{1,2,3\}) - m(\{1,2\}))+\\ &\overline{x}_2(m(\{1,2\}) - m(\{1\}))+\\ &\overline{x}_1 m(\{1\}).\end{aligned}$$

Since $(m(\{1,2,3\}) - m(\{1,2\})) = 0.5$, $\overline{x}_3$ gets more importance in the fused output, thus explaining the behavior of $\mathbf{C}_m$ in contrast with $\mathbf{C}_m^{arith}$.

Now, we prove that our proposed integral extends the standar Choquet integral by considering degenerated intervals (numbers).

Proposition 3: It holds that $\mathbf{C}_m^{arith}([x_1, x_1], \dots, [x_n, x_n]) = [C_m(x_1, \dots, x_n), C_m(x_1, \dots, x_n)]$ for every $x_1, \dots, x_n \in [0, 1]$.

Proof: Notice that if there is a unique admissible permutation, then the result is clear. Otherwise, observe that $\mathbf{C}_m^{\sigma_1}([x_1, x_1], \dots, [x_n, x_n]) = \cdots = \mathbf{C}_m^{\sigma_p}([x_1, x_1], \dots, [x_n, x_n]) = [C_m(x_1, \dots, x_n), C_m(x_1, \dots, x_n)]$, being $\sigma_1, \dots, \sigma_p$ the set of admissible permutatinons of $([x_1, x_1], \dots, [x_n, x_n])$ and the result follows. ∎

The next proposition shows interesting behaviors of $\mathbf{C}_m^{arith}$ when certain input vectors are considered. The first result shows that if every interval considered can be compared by means of $\leq_L$, then we obtain Auman's approach. In the opposite way, if none of the intervals can be compared (they form an antichain), then no order structure can be obtained and the integral yields in the arithmetic mean.

Proposition 4: The following items hold:

(i) if $\mathbf{x}_1, \dots, \mathbf{x}_n$ form a chain, then $\mathbf{C}_m^{arith}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{C}_m(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{C}_m^{\preceq_{A,B}}$ for every admissible order $\preceq_{A,B}$.

(ii) if $\mathbf{x}_1, \dots, \mathbf{x}_n$ form an antichain, then for any fuzzy measure m, $\mathbf{C}_m^{arith}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{M}_{arith}(\mathbf{x}_1, \dots, \mathbf{x}_n)$.

Proof: Direct. ∎

We now investigate whether our new approach is an interval aggregation function and which properties of aggregation functions are satisfied. The next proposition shows that our approach is idempotent and bounded by the infimum and supremum of inputs, two main properties satisfied by *averaging functions* or *means* [2].

Proposition 5: The following items hold:

(i) $\mathbf{C}_m^{arith}(\mathbf{x}, \dots, \mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in L([0, 1])$;

(ii) $\inf\{\mathbf{x}_1, \dots, \mathbf{x}_n\} \leq_L \mathbf{C}_m^{arith}(\mathbf{x}_1, \dots, \mathbf{x}_n) \leq_L \sup\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ for every $\mathbf{x}_1, \dots, \mathbf{x}_n \in L([0, 1])$;

Proof: (i) Although there exist $n!$ admissible permutations of $(\mathbf{x}, \dots, \mathbf{x})$, we have that $\mathbf{C}_m^{\sigma_1}(\mathbf{x}, \dots, \mathbf{x}) = \cdots = \mathbf{C}_m^{\sigma_{n!}}(\mathbf{x}, \dots, \mathbf{x}) = \mathbf{x}$, since $\mathbf{M}_{arith}$ is idempotent. (ii) Boundedness holds since $\mathbf{C}_m^{\sigma}$ and $\mathbf{M}_{arith}$ are also bounded by infimum and supremum of inputs. ∎

However, we cannot consider $\mathbf{C}_m^{arith}$ as an interval aggregation functions since the monotonicity is not satisfied. This fact is herited from the same possible failure of $\mathbf{C}_m^{\preceq_{A,B}}$ integrals.

Example 7: Let $n = 3$ and $\mathbf{x}_1 = [0.2, 0.9], \mathbf{x}_2 = [0.3, 0.7]$ and $\mathbf{x}_3 = [0.5, 0.6]$. Let $m = m_*$ (bottom fuzzy measure). Since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ form an antichain, then $\mathbf{C}_m^{arith}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbf{M}_{arith}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \left[\frac{1}{3}, \frac{11}{15}\right]$. Suppose now that we increase $\mathbf{x}_1$ and $\mathbf{x}_3$ to $\mathbf{x}'_1 = \mathbf{x}'_3 = [1, 1]$. Now, $\mathbf{x}'_1, \mathbf{x}_2, \mathbf{x}'_3$ form a chain and $\mathbf{C}_m^{arith}(\mathbf{x}'_1, \mathbf{x}_2, \mathbf{x}'_3) = \mathbf{C}_m(\mathbf{x}'_1, \mathbf{x}_2, \mathbf{x}'_3) = [0.3, 0.7]$. Note that, while $\mathbf{x}_1 \leq_L \mathbf{x}'_1$ and $\mathbf{x}_3 \leq_L \mathbf{x}'_3$, we have that $[0.3, 0.7] \leq_L \left[\frac{1}{3}, \frac{11}{15}\right]$ and monotonicity is not satisfied.

Even if $\mathbf{C}_m^{arith}$ does not fulfill monotonicity, new trends in aggregation functions are introducing new types of monotonicities. This is the case of weak monotonicity [24] or directional monotonicity [25]. In fact, our new proposal can be seen as a weak monotone function or a directional monotone function with respect to $r = (1, \ldots, 1)$, since $\mathbf{C}_m^{arith}$ is invariant under translation.

Lemma 1: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$ and let $\mathbf{k}, \mathbf{r} \in L([0,1])$ such that $\mathbf{k}\,\mathbf{x}_1 + \mathbf{r}, \ldots, \mathbf{k}\,\mathbf{x}_n + \mathbf{r} \in L([0,1])$, $\mathbf{k} > 0_L$ ($\mathbf{k}$ is a closed subinterval of $]0,1]$) . Let Σ be the set of admissible permutations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and let Σ' be the set of admissible permutations of $\mathbf{k}\,\mathbf{x}_1 + \mathbf{r}, \ldots, \mathbf{k}\,\mathbf{x}_n + \mathbf{r}$. Then, $\Sigma = \Sigma'$.

Proof: Let $\sigma \in \Sigma$ and consider some $i, j \in \{1, \ldots, n\}$ with $\mathbf{k}\,\mathbf{x}_{\sigma(i)} + \mathbf{r} <_L \mathbf{k}\,\mathbf{x}_{\sigma(j)} + \mathbf{r}$. Then, it must be $\mathbf{x}_{\sigma(i)} <_L \mathbf{x}_{\sigma(j)}$. Since $\sigma \in \Sigma$, this implies $i < j$. Therefore, $\sigma \in \Sigma'$. The same reasoning can be done for $\sigma \in \Sigma'$, so we have that $\Sigma = \Sigma'$. If there is not i, j satisfying $\mathbf{x}_{\sigma(i)} + \mathbf{r} <_L \mathbf{k}\,\mathbf{x}_{\sigma(j)} + \mathbf{r}$, then either $\mathbf{x}_1 = \cdots = \mathbf{x}_n$ or $\mathbf{x}_1, \ldots, \mathbf{x}_n$ form an antichain. In both cases, it is easy to see that $\Sigma = \Sigma'$. ∎

Proposition 6: The following items hold:

(i) $\mathbf{C}_m^{arith}(\mathbf{k}\,\mathbf{x}_1, \ldots, \mathbf{k}\,\mathbf{x}_n) = \mathbf{k}\,\mathbf{C}_m^{arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ for every $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{k} \in L([0,1])$;

(ii) $\mathbf{C}_m^{arith}(\mathbf{x}_1 + \mathbf{r}, \ldots, \mathbf{x}_n + \mathbf{r}) = \mathbf{C}_m^{arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \mathbf{r}$ for every $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{r} \in L([0,1])$ with $\mathbf{x}_1 + \mathbf{r}, \ldots, \mathbf{x}_n + \mathbf{r} \in L([0,1])$.

Proof: By Lemma 1, we know that the set of admissible permutations are the same. The result follows from the fact that the arithmetic mean is stable under any positive linear transformation. ∎

Notice it is not easy to establish an order relation between $\mathbf{C}_m^{arith}$ and other approaches given in Section III. However, since $\mathbf{C}_m^{inf}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m^{arith}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m^{sup}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, we have also the following result.

Proposition 7: For every fuzzy measure m and every $\mathbf{x}_1, \ldots, \mathbf{x}_n \in L([0,1])$, it holds that

$$\mathbf{C}_m^{\inf}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m^{\sup}(\mathbf{x}_1, \ldots, \mathbf{x}_n).$$

Proof: If we consider the lowerbound, just observe that $\underline{C}_m(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \underline{C}_m^{Lex1}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The same reasoning can be done for the upper bounds considering that $\overline{C}_m(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \overline{C}_m^{Lex2}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. ∎

Proposition 7 can be easily extended for every admissible order $\preceq_{\mathrm{A,B}}$, since $\mathbf{C}_m^{inf}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m^{\preceq_{\mathrm{A,B}}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_L \mathbf{C}_m^{sup}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

VI. An Application of Admissible Permutations and the Interval-Valued Choquet Integral in Decision Making

In multicriteria decision making problems, the objective is to find the most preferred alternative over a set of p alternatives $A = \{a_1, \ldots, a_p\}$ with respect to a set of criteria. We assume that the expert evaluates their preference between each pair of alternatives. Since expressing these preferences with an exact value may be difficult or problematic for he/she, we allow the expert to express the preference of one alternative a_i against another a_j (with $i, j \in \{1, \ldots, p\}$) by means of an interval, i.e. elements of $L([0,1])$. The collection of preferences of every pair of alternatives is done by means of an interval-valued fuzzy relation $\mathbf{R} : A \times A \to L([0,1])$ (see [26]). The interval $\mathbf{R}(a_i, a_j)$ represents the degree to which the alternative a_i is preferred to alternative a_j. For this reason, we also denote $\mathbf{R}$ as an interval-valued fuzzy preference relation or IVFPR.

Once the preference relation is constructed, it must be exploited to obtain a final evaluation of each alternative. To do this, we will consider an interval fusion function $\mathbf{F_a}$ that will by applied to each row of the IVFPR. This operator will aggregate, for each alternative, its preferences, so the higher the value, the more preferred the alternative. Then, for each alternative a_i, we will obtain a final evaluation P_{a_i} which is given by

$$\begin{aligned} P_{a_i} = \mathbf{F_a}(\,&\mathbf{R}(a_i, a_1), \ldots, \mathbf{R}(a_i, a_{i-1}), \\ &\mathbf{R}(a_i, a_{i+1}), \ldots, \mathbf{R}(a_i, a_p)), \end{aligned}$$

where $\mathbf{R}$ is an IVFPR and $\mathbf{F_a} : L([0,1])^{p-1} \to L([0,1])$ is an interval fusion function.

As we have commented in this paper, the usual partial order of intervals $\leq_L$ may not be useful in order to obtain the final decision, i.e. the best alternative or an ordered sequence of alternatives (ordered by preference). This is due to the fact that the interval evaluations obtained $(P_{a_1}, \ldots, P_{a_p})$ may not be comparable. For solving this problem, we propose a simple rule to obtain the best alternative which is based on the set of admissible permutations of $P_{a_1}, \ldots, P_{a_p}$. For each admissible permutation σ, each alternative gets the same number of votes as its relative position in the ordered sequence of alternatives (by means of σ). Then, $a_{\sigma(1)}$ will get one vote, $a_{\sigma(2)}$ will get two votes, and so on. After computing every admissible permutation, the best alternative is selected as the one with higher number of votes

The whole decision-making algorithm is summarized in Algorithm 1.

Algorithm 1 Decision Making based on admissible permutations

Require: Set of alternatives $A = \{a_1, \ldots, a_p\}$;
  IVPFR $\mathbf{R} : A \times A \to L([0,1])$;
  Interval fusion function $\mathbf{F} : L([0,1])^{p-1} \to L([0,1])$.
Ensure: Best alternative a^*
```
for i = 1, ..., p do
    P_{a_i} = F_a^n_{j=1, j≠i} (R_C(a_i, a_j))
end for
Set votes V_{a_1}, ..., V_{a_p} = 0
Calculate the set of admissible permutations σ_1, ..., σ_m
    of P_{a_1}, ..., P_{a_p}
for i = 1, ..., m do
    for j = 1, ..., p do
        V_{a_{σ_i(j)}} := V_{a_{σ_i(j)}} + j
    end for
end for
Assign a_* = arg max_i V_{a_i} (if there is a tie, then choose
    arbitrary one of the tied alternative)
```

Example 8: The following example have been taken from [27], where four suppliers $a_1, \ldots, a_4$ are evaluated taking into account environmental criteria. The expert has provided the following IVFPR:

$$\begin{pmatrix} - & [0.35,\, 0.45] & [0.50, 0.70] & [0.40, 0.50] \\ [0.55, 0.65] & - & [0.60, 0.80] & [0.20, 0.60] \\ [0.30, 0.50] & [0.20, 0.40] & - & [0.40, 0.60] \\ [0.50, 0.60] & [0.40, 0.80] & [0.40, 0.60] & - \end{pmatrix}$$

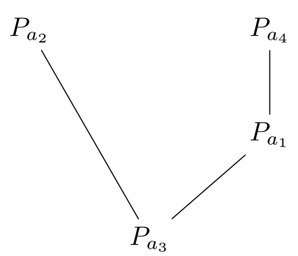
Consider now the following fuzzy measure $m : 2^{\{1,2,3\}} \to [0, 1]$ given by:

$$\begin{array}{lll} m(\emptyset) = 0 & & \\ m(\{1\}) = 0.3 & m(\{2\}) = 0.1 & m(\{3\}) = 0.35 \\ m(\{1, 2\}) = 0.3 & m(\{1, 3\}) = 0.5 & m(\{2, 3\}) = 0.8 \\ m(\{1, 2, 3\}) = 1 & & \end{array}$$

Now, we apply $\mathbf{C}_m$ to every row of the IVFPR obtaining the following evaluations for each alternative:

$$\begin{aligned} P_{a_1} &= [0.400, 0.510], \\ P_{a_2} &= [0.310, 0.630], \\ P_{a_3} &= [0.285, 0.485], \\ P_{a_4} &= [0.420, 0.620]. \end{aligned}$$

Observe that $P_{a_3} \leq_L P_{a_i} fori \in \{1, 2, 4\}$ and that $P_{a_1} \leq_L P_{a_4}$, but the rest of intervals cannot be compared. This is illustrated in the following Hasse diagram:



The set of admissible permutations of $P_{a_1}, \ldots, P_{a_4}$ is composed by $m = 3$ permutations, namely $\sigma_1, \sigma_2, \sigma_3 : \{1, \ldots, 4\} \to \{1, \ldots, 4\}$ which are given by:

$$\begin{aligned} \sigma_1 &= (3, 1, 2, 4) \\ \sigma_2 &= (3, 1, 4, 2) \\ \sigma_3 &= (3, 2, 1, 4) \end{aligned}$$

Finally, let us calculate the votes for each alternative based on $\sigma_1, \sigma_2, \sigma_3$, which are shown in the following table:

	V_{a_1}	V_{a_2}	V_{a_3}	V_{a_4}
σ_1	2	3	1	4
σ_2	2	4	1	3
σ_3	3	2	1	4
	7	9	3	11

This means that $V_{a_1} = 7$, $V_{a_2} = 9$, $V_{a_3} = 3$ and $V_{a_4} = 11$, so we can order the alternatives with the following preference

$$a_3 \prec a_1 \prec a_2 \prec a_4.$$

VII. Conclusions

In this work we have proposed a new fusion mechanism for interval data based on the Choquet integral that considers the ordinal structure of data in the fusion process. Although there exist some other approaches to the integration of interval/intuitionistic data (see for example [28], [29], [30], [31], [32], [33], [34], [35] for some examples in the intuitionistic framework), they do not extend the standard Choquet integral. That is, when we consider real numbers, we expect these extensions to recover the result of the original Choquet integral of real inputs.

Our approach, that recovers the original definition when considering real numbers, is based on a consensus (or agreement) between interval-valued Choquet integrals constructed with different settings. In order to do so, we have first proposed the concept of an admissible permutation. This has allowed to calculate the different number of arrangements (satisfying some basic properties) of the input data. Then, the result of each interval-valued Choquet integral based on a specific permutation is finally fused into a single interval output that models the consensus among the different permutations.

The proposed fusion procedure focuses on the Choquet integral. However, it is easy to see that it can be extended to those fusion functions based on the arrangement of data, such as the OWA operators or the Sugeno integral. Moreover, we think that it would be interesting to study the importance of each individual permutation, allowing therefore to calculate the consensus by means of weighted functions or some other functions that simulate the attitude of the final user.

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